

EIGENVALUES OF THE BILAYER GRAPHENE OPERATOR WITH A COMPLEX VALUED POTENTIAL

FRANCESCO FERRULLI, ARI LAPTEV AND OLEG SAFRONOV

ABSTRACT. We study the spectrum of a system of second order differential operator D_m perturbed by a non-selfadjoint matrix valued potential V . We prove that eigenvalues of $D_m + V$ are located near the edges of the spectrum of the unperturbed operator D_m .

1. STATEMENT OF THE MAIN RESULTS

Spectral properties of non-selfadjoint operators have been recently a subject of interest of many papers. A particular interest was related to the location of eigenvalues of differential operators in the complex plane \mathbb{C} . The corresponding results for Schrödinger operators can be found in [1], [3]-[4] and in [5]. Some other problems were studied in the papers [6]-[10] and [12].

The operator we study is related to the quantum theory of a material consisting of two layers of graphene. Namely, we consider the operator $D = D_m + V$, where

$$D_m = \begin{pmatrix} m & 4\partial_{\bar{z}}^2 \\ 4\partial_z^2 & -m \end{pmatrix}, \quad \partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x_1} + \frac{1}{i} \frac{\partial}{\partial x_2} \right), \quad \partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x_1} - \frac{1}{i} \frac{\partial}{\partial x_2} \right), \quad m \geq 0.$$

This operator acts in the Hilbert space $L^2(\mathbb{R}^2; \mathbb{C}^2)$. The domain of D is the Sobolev space $\mathcal{H}^2(\mathbb{R}^2; \mathbb{C}^2)$. The potential V is a not necessary self-adjoint matrix-valued function

$$V(x) = \begin{pmatrix} V_{1,1}(x) & V_{1,2}(x) \\ V_{2,1}(x) & V_{2,2}(x) \end{pmatrix}$$

where the matrix elements are allowed to take complex values. For the matrix V we denote

$$|V(x)| = \sqrt{\sum_{i,j=1,2} |V_{i,j}(x)|^2}.$$

Assuming that V decays at the infinity in some integral sense we would like to answer the question: "Where are the eigenvalues of D located?"

Note that since $D_m^2 = \Delta^2 + m^2$, the spectrum $\sigma(D_m)$ of D_m is the set $(-\infty, m] \cup [m, \infty)$. Our results show that the eigenvalues of D are located near the edges of the absolutely continuous spectrum, i.e. near the points $\pm m$. Since the spectrum of the unperturbed operator has two edges, our results resemble some of the theorems of the paper [2] related to the Dirac operator. However, the main difference between the two papers is that we study a differential operator on a plane, while the article [2] deals with operators on a line.

Theorem 1.1. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator D . Let $1 < p < 4/3$. Then*

$$\frac{C_p \int_{\mathbb{R}^2} |V(x)|^p dx}{|\mu|^{p-1}} \left(\sqrt{\left| \frac{k-m}{k+m} \right|} + \sqrt{\left| \frac{k+m}{k-m} \right|} + 1 \right)^p \geq 1, \quad \mu^2 = k^2 - m^2,$$

with $C_p > 0$ independent of V , k and m . In particular, if $m = 0$, then

$$|k|^{p-1} \leq 3^p C_p \int_{\mathbb{R}^2} |V(x)|^p dx, \quad 1 < p < 4/3.$$

The next statement tells us about what happens when $p \rightarrow 1$.

Theorem 1.2. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator D . Let $\mu^2 = k^2 - m^2$. Then*

$$\begin{aligned} & C \left(|\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y| < (2|\mu|)^{-1}} |V(y)| dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \left(1 + |\ln |x-y|| \right) |V(y)| dy \right) + \\ & + C \int_{\mathbb{R}^2} |V(x)| dx \left(\sqrt{\left| \frac{k-m}{k+m} \right|} + \sqrt{\left| \frac{k+m}{k-m} \right|} + 1 \right) \geq 1, \end{aligned}$$

where the constant $C > 0$ is independent of V , k and m .

Note that this statement also holds true for $m = 0$.

Corollary 1.1. *Let $m = 0$ and let $k \notin \mathbb{R}$ be an eigenvalue of the operator D . Then*

$$\begin{aligned} & C \left(|\ln |k|| \sup_{x \in \mathbb{R}^2} \int_{|x-y| < (2|k|)^{-1}} |V(y)| dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \left(1 + |\ln |x-y|| \right) |V(y)| dy \right) + \\ & + 3C \int_{\mathbb{R}^2} |V(x)| dx \geq 1, \end{aligned}$$

where the constant $C > 0$ is independent of V and k .

In particular, we see that if $m = 0$, then for small V , the eigenvalues of D are situated in the circle $\{k \in \mathbb{C} : |k| < r\}$ of radius r which has the following asymptotical behavior

$$r \asymp \exp\left(-\frac{C}{\int |V| dx}\right), \quad \text{as} \quad \int |V| dx \rightarrow 0.$$

The proof of Theorems 1.1 and 1.2 are given in Section 2. In Section 3 we consider a special case where $V = iW^2$, $W = W^*$. In this case we can get a more precise information about location of the complex eigenvalues, see Theorem 3.1. It is interesting to note that if $m = 0$ (no gap in the continuous spectrum), then perturbations by such matrix-functions do not create any complex eigenvalues. Here we have similarities with the result obtained for the one dimensional Dirac operators in [2].

2. PROOFS OF THE MAIN RESULTS

In order to prove our main results we need the Birman-Schwinger principle formulated below.

Proposition 2.1. *Let $V = W_2 W_1$, where W_1 and W_2 are two matrix-valued decaying functions. A point $k \in \mathbb{C} \setminus \sigma(D_m)$ is an eigenvalue of D if and only if -1 is an eigenvalue of the operator*

$$X(k) := W_1(D_m - k)^{-1} W_2.$$

In particular, if $k \in \mathbb{C} \setminus \sigma(D_m)$ is an eigenvalue of D then $\|X(k)\| \geq 1$.

The proof of this statement is standard and it is left to the reader as an exercise.

Below we always denote

$$W = \sqrt{V^*V}$$

and use the Birman-Schwinger principle with $W_1 = W$ and $W_2 = VW^{-1/2}$.

Proof of Theorem 1.1. Since

$$(D_m - k)^{-1} = (D_m + k)(D_m - k)^{-1}(D_m + k)^{-1} = (D_m + k)(D_m^2 - k^2)^{-1},$$

it is easy to see that

$$(1) \quad (D_m - k)^{-1} = (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1},$$

where

$$D_0 = \begin{pmatrix} 0 & 4\partial_z^2 \\ 4\partial_z^2 & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can also note that the last term in the right hand side of (1) can be rewritten in the form

$$(2) \quad (D_0 - \mu)^{-1} = (D_0 + \mu)(\Delta^2 - \mu^2)^{-1}.$$

The operator $(\Delta^2 - \mu^2)^{-1}$ is an integral operator with the kernel

$$g_k(x, y) = \frac{i}{8\mu} \left(H(\sqrt{\mu}r) - H(i\sqrt{\mu}r) \right),$$

where $H(z) = H_0^{(1)}(z)$ is the Hankel function of first kind and $r = |x - y|$. It is a simple consequence of the fact that

$$(\Delta^2 - \mu^2)^{-1} = \frac{1}{2\mu} \left((-\Delta - \mu)^{-1} - (-\Delta + \mu)^{-1} \right).$$

The kernel of $(-\Delta - \mu)^{-1}$ is $4^{-1}iH(\sqrt{\mu}r)$. Another useful representation of $g_k(x, y)$ follows from the fact that the kernel of $(-\Delta - \mu)^{-1}$ equals (see [11])

$$(2\pi)^{-1}K_0(-i\sqrt{\mu}|x - y|),$$

where

$$K_0(z) = \frac{e^{-z}}{\Gamma(1/2)} \sqrt{\frac{\pi}{2z}} \int_0^\infty e^{-t} t^{-1/2} \left(1 + \frac{t}{2z}\right)^{-1/2} dt, \quad |\arg z| < \pi.$$

Let us define

$$G(z) = H(z) - H(iz).$$

We need to know the behaviour of the function G only in the region $0 < \arg z < \pi/2$, where we have

$$|G(z)| + |G'(z)| + |G''(z)| \leq \frac{C}{\sqrt{|z|}}, \quad \text{if } |z| > 1/2.$$

The behaviour of the function G near zero is determined by the expansion of the Hankel function in the neighbourhood of $z = 0$. It turns out that

$$|G(z)| \leq C, \quad |G'(z)| \leq C_1 |z| \ln |z|^{-1}, \quad |G''(z)| \leq C_1 \ln |z|^{-1}, \quad \text{if } |z| < 1/2.$$

Let $\rho_\mu(|x - y|)$ be the kernel of the integral operator $(D_0 - \mu)^{-1}$

$$\rho_\mu(|x - y|) = \frac{i}{8\mu} \begin{pmatrix} \mu G(\sqrt{\mu}|x - y|) & \partial_z^2 G(\sqrt{\mu}|x - y|) \\ \partial_z^2 G(\sqrt{\mu}|x - y|) & \mu G(\sqrt{\mu}|x - y|) \end{pmatrix}$$

Therefore

$$|\rho_\mu(|x - y|)| = \frac{1}{8|\mu|} \sqrt{2|\mu|^2 |G(\sqrt{\mu}|x - y|)|^2 + |\partial_{\bar{z}}^2 G(\sqrt{\mu}|x - y|)|^2 + |\partial_z^2 G(\sqrt{\mu}|x - y|)|^2}.$$

As a consequence, if we denote by $\rho_\theta(|x - y|)$ the kernel of the operator $(D_0 - e^{i\theta})^{-1}$ then

$$(3) \quad |\rho_\theta(r)| \leq C \ln r^{-1}, \quad \text{if} \quad r < 1/2,$$

and

$$(4) \quad |\rho_\theta(r)| \leq C r^{-1/2}, \quad \text{if} \quad r > 1/2.$$

In order to prove the latter relations, one has to differentiate the integral kernel of $(\Delta^2 - \mu^2)^{-1}$, using the formulas

$$\frac{\partial r}{\partial z} = \frac{1}{2} \frac{\bar{z}}{r}, \quad \frac{\partial^2 r}{\partial z^2} = -\frac{1}{4} \frac{\bar{z}^2}{r^3}$$

and

$$\frac{\partial r}{\partial \bar{z}} = \frac{1}{2} \frac{z}{r}, \quad \frac{\partial^2 r}{\partial \bar{z}^2} = -\frac{1}{4} \frac{z^2}{r^3}.$$

Since the integral kernel of $(\Delta^2 - \mu^2)^{-1}$ is $\frac{i}{8\mu} G(\sqrt{\mu}r)$, we obtain from (2) that

$$\begin{aligned} 8|\rho_\theta(r)| &\leq \left(\left| \frac{\partial^2 G(e^{i\theta/2}r)}{\partial z^2} \right|^2 + \left| \frac{\partial^2 G(e^{i\theta/2}r)}{\partial \bar{z}^2} \right|^2 + 2|G(e^{i\theta/2}r)|^2 \right)^{1/2} \\ &\leq C(r^{-1}|G'(e^{i\theta/2}r)| + |G''(e^{i\theta/2}r)| + |G(e^{i\theta/2}r)|). \end{aligned}$$

The positive constants in the inequalities (3) and (4) do not depend on $\theta \in [0, \pi/2]$. In particular,

$$M := \sup_{\theta} \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |\rho_\theta(|x - y|)|^q dy < \infty, \quad q > 4.$$

Let us estimate now the norm of the operator $T = W(D_0 - e^{i\theta})^{-1}W$ with the kernel

$$\tau(x, y) = W(x)\rho_\theta(|x - y|)W(y).$$

For that purpose, we estimate the sesquie-linear form of this operator :

$$(Tu, v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x)W(x)\rho_\theta(|x - y|)W(y)u(y) dx dy.$$

Obviously,

$$\begin{aligned} |(Tu, v)|^2 &= \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x)W(x)\rho_\theta(|x - y|)W(y)u(y) dx dy \right|^2 \leq \\ &\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v(x)|^2 |\rho_\theta(|x - y|)| |W(y)|^2 dx dy \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W(x)|^2 |\rho_\theta(|x - y|)| |u(y)|^2 dx dy \leq \\ &\left(\sup_x \int_{\mathbb{R}^2} |\rho_\theta(|x - y|)| |W(y)|^2 dy \right)^2 \|u\|^2 \|v\|^2 \leq \\ &\left(\int_{\mathbb{R}^2} |\rho_\theta(|x - y|)|^q dy \right)^{2/q} \|V\|_p^2 \|u\|^2 \|v\|^2, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad q > 4. \end{aligned}$$

Therefore,

$$\|T\| \leq C \|V\|_p, \quad 1 < p < 4/3.$$

We are now able to estimate the norm of the operator $T_k = W(D_0 - k)^{-1}W$ for $k \notin \sigma(D_0)$. Indeed,

$$|(T_k u, v)| = \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x) W(x) \rho_\theta(\sqrt{|k|}|x - y|) W(y) u(y) dx dy \right| =$$

$$\frac{1}{|k|^2} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x/\sqrt{|k|}) W(x/\sqrt{|k|}) \rho_\theta(|x - y|) W(y/\sqrt{|k|}) u(y/\sqrt{|k|}) dx dy \right| \leq$$

$$\frac{C}{|k|^2} \|V(\cdot/\sqrt{|k|})\|_p \|u(\cdot/\sqrt{|k|})\| \|v(\cdot/\sqrt{|k|})\| = \frac{C\|V\|_p}{|k|^{(p-1)/p}} \|u\| \|v\|.$$

Consequently,

$$\|T_k\| \leq \frac{C\|V\|_p}{|k|^{(p-1)/p}}.$$

Observe now that the kernel of the operator $(\Delta^2 - \mu^2)^{-1}$ is the function $iG(\sqrt{\mu}|x - y|)/(8\mu)$. The function $G(\sqrt{\mu}|x - y|)$ has the same properties as $\rho_\theta(\sqrt{|\mu|}|x - y|)$. Moreover it is bounded. Therefore, by mimicking the above arguments, one proves that

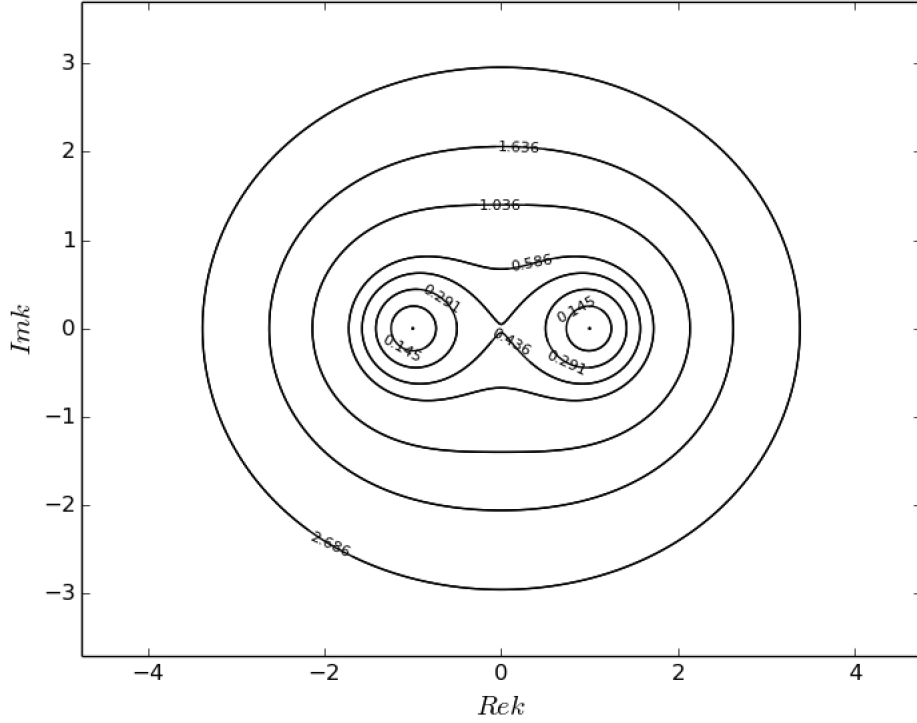
$$(5) \quad \|W(\Delta^2 - \mu^2)^{-1}W\| \leq \frac{C\|V\|_p}{|\mu|^{(2p-1)/p}}, \quad 1 \leq p < 4/3.$$

This leads to the estimate

$$\|W(D_m - k)^{-1}W\|^p \leq \frac{C \int_{\mathbb{R}^2} |V(x)|^p dx}{|\mu|^{p-1}} \left(\sqrt{\left| \frac{k+m}{k-m} \right|} + \sqrt{\left| \frac{k-m}{k+m} \right|} + 1 \right)^p, \quad \mu^2 = k^2 - m^2.$$

Now the statement of our theorem follows from the fact that if k is an eigenvalue of $D = D_m + V$, then $\|W(D_m - k)^{-1}W\| \geq 1$. The proof is complete.

In the picture below we describe the areas of possible location of complex eigenvalues depending on the value of $C \int_{\mathbb{R}^2} |V(x)|^p dx$, where $m = 1$ and $p = 1.2$.



Proof of Theorem 1.2. As before we use the representation

$$(D_m - k)^{-1} = (m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} + (D_0 - \mu)^{-1}.$$

The operator $(\Delta^2 - \mu^2)^{-1}$ is an integral operator with the kernel

$$g_k(x, y) = \frac{i}{8\mu} \left(H(\sqrt{\mu}r) - H(i\sqrt{\mu}r) \right),$$

where $H(z) = H_0^{(1)}(z)$ is the Hankel function of first kind. Again we denote

$$G(z) = H(z) - H(iz).$$

We need to know the behavior of the function G only in the region $0 < \arg z < \pi/2$, where this function is bounded. The boundedness of G implies the estimate (5) with $p = 1$.

It remains to estimate the norm of the operator $T_\mu = W(D_0 - \mu)^{-1}W$ for $\text{Im } \mu > 0$. We already know that if $\mu = |\mu|e^{i\theta}$ the operator $(D_0 - \mu)^{-1}$ is an integral operator with the kernel $\rho_\theta(\sqrt{|\mu|}|x - y|)$, where ρ_θ is a function having the properties

$$(6) \quad |\rho_\theta(r)| \leq C \ln r^{-1}, \quad \text{if} \quad r < 1/2,$$

and

$$(7) \quad |\rho_\theta(r)| \leq Cr^{-1/2}, \quad \text{if} \quad r > 1/2.$$

The positive constants in these inequalities do not depend on $\theta \in [0, \pi]$. As before, we estimate the sesquie-linear form of this operator :

$$(T_\mu u, v) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x) W(x) \rho_\theta(\sqrt{|\mu|}|x - y|) W(y) u(y) dx dy.$$

Obviously,

$$\begin{aligned} |(T_\mu u, v)|^2 &= \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \bar{v}(x) W(x) \rho_\theta(\sqrt{|\mu|}|x-y|) W(y) u(y) dx dy \right|^2 \leq \\ &\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v(x)|^2 |\rho_\theta(\sqrt{|\mu|}|x-y|)| |W(y)|^2 dx dy \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W(x)|^2 |\rho_\theta(\sqrt{|\mu|}|x-y|)| |u(y)|^2 dx dy \leq \\ &\left(\sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{|\mu|}|x-y|)| |V(y)| dy \right)^2 \|u\|^2 \|v\|^2. \end{aligned}$$

Therefore,

$$\|T_\mu\| \leq \sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{|\mu|}|x-y|)| |V(y)| dy.$$

The bounds (6) and (7) imply

$$\begin{aligned} &\sup_x \int_{\mathbb{R}^2} |\rho_\theta(\sqrt{|\mu|}|x-y|)| |V(y)| dy \\ &\leq C \left(|\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y| < (2|\mu|)^{-1}} |V(y)| dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} (1 + |\ln |x-y||) |V(y)| dy \right), \end{aligned}$$

which leads to

$$\|T_\mu\| \leq C \left(|\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y| < (2|\mu|)^{-1}} |V(y)| dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} (1 + |\ln |x-y||) |V(y)| dy \right).$$

Since

$$\|W(D_m - k)^{-1} W\| \leq \|W(m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1} W\| + \|T_\mu\|$$

and since (5) holds with $p = 1$, we obtain

$$\begin{aligned} &\|W(D_m - k)^{-1} W\| \\ &\leq C \left(|\ln |\mu|| \sup_{x \in \mathbb{R}^2} \int_{|x-y| < (2|\mu|)^{-1}} |V(y)| dy + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} (1 + |\ln |x-y||) |V(y)| dy \right) \\ &\quad + C \int_{\mathbb{R}^2} |V(x)| dx \left(\sqrt{\left| \frac{kt - m}{k + m} \right|} + \sqrt{\left| \frac{k + m}{k - m} \right|} + 1 \right). \end{aligned}$$

The statement of Theorem 1.2 follows from the fact that if k is an eigenvalue of $D = D_m + V$, then $\|W(D_m - k)^{-1} W\| \geq 1$.

3. A SPECIAL CASE

Consider now a special case, when $V(x) = iW^2(x)$, where $W(x) = W^*(x)$ is a matrix valued function. It turns out, that in this case we can get a more precise information about the spectral properties of the operator D .

Theorem 3.1. *Let $k \notin \sigma(D_m)$ be an eigenvalue of the operator $D = D_m + V$, where $V = iW^2$. Let μ be the number in the upper half-plane defined by $\mu^2 = k^2 - m^2$. Then*

$$(8) \quad \left(C \left(\left| \frac{k + m}{\mu} - 1 \right| + \left| \frac{k - m}{\mu} - 1 \right| \right) + 1 \right) \frac{1}{4} \int_{\mathbb{R}^2} \text{tr} |V| dx \geq 1,$$

where the constant C is independent of V , m and k .

Proof. According to the Birman-Schwinger principle, k is an eigenvalue of the operator D if and only if 1 is an eigenvalue of the operator $X = -iW(D_m - k)^{-1}W$. On the other hand, if 1 is an eigenvalue of X then $\|\operatorname{Re} X\| \geq 1$. Since,

$$\operatorname{Re} X = W \operatorname{Im}(D_m - k)^{-1}W,$$

we would like to have the explicit expression for the operator $\operatorname{Im}(D_m - k)^{-1}$. Let us first obtain this representation for the case $m = 0$. Observe that D_0 is the operator with the symbol

$$\begin{pmatrix} 0 & -(\xi_1 + i\xi_2)^2 \\ -(\xi_1 + i\xi_2)^2 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\pm|\xi_1 + i\xi_2|^2$. The orthogonal projections $P_1(\xi)$ and $P_2(\xi)$, $\xi = \xi_1 + i\xi_2$, onto the eigenvectors depend only on $\arg(\xi)$. Therefore, the symbol of D_0 is

$$|\xi|^2 P_1(\xi) - |\xi|^2 P_2(\xi),$$

which implies that the integral kernel of $\operatorname{Im}(D_0 - k)^{-1}$ is

$$(2\pi)^{-2} \int_{\mathbb{R}^2} \exp(i\xi(x - y)) \left(\frac{(\operatorname{Im} k) P_1(\xi)}{(|\xi|^2 - \operatorname{Re} k)^2 + (\operatorname{Im} k)^2} + \frac{(\operatorname{Im} k) P_2(\xi)}{(-|\xi|^2 - \operatorname{Re} k)^2 + (\operatorname{Im} k)^2} \right) d\xi.$$

It follows from this representation that the kernel of the operator $\operatorname{Im}(D_0 - k)^{-1}$ is bounded by $1/4$ as using polar coordinates and changing variables $|\xi|^2 = t$ we obtain

$$\|\operatorname{Im}(D_0 - k)^{-1}\| \leq (2\pi)^{-2} \int_{\mathbb{S}^1} \int_{-\infty}^{\infty} \frac{\operatorname{Im} k}{t^2 + (\operatorname{Im} k)^2} dt = \frac{1}{4}, \quad \operatorname{Im} k > 0.$$

Consequently,

$$\|W \operatorname{Im}(D_0 - k)^{-1}W\| \leq \operatorname{tr}(W \operatorname{Im}(D_0 - k)^{-1}W) \leq \frac{1}{4} \int_{\mathbb{R}^2} \operatorname{tr} W^2(x) dx.$$

If $m > 0$, then we have

$$\|\operatorname{Re} X\| \leq \left\| \frac{1}{2\mu} W(m\gamma_0 + k - \mu)(\Delta^2 - \mu^2)^{-1}W \right\| + \|W \operatorname{Im}(D_0 - \mu)^{-1}W\|$$

and that

$$\left\| \frac{m\gamma_0 + k - \mu}{2\mu} \right\| \leq \frac{1}{2} \left(\left| \frac{k + m}{\mu} - 1 \right| + \left| \frac{k - m}{\mu} - 1 \right| \right).$$

It remains to note that according to (5) with $p = 1$,

$$\|W(\Delta^2 - \mu^2)^{-1}W\| \leq C \int_{\mathbb{R}^2} \operatorname{tr} |V| dx$$

The proof is completed. \square

The next result says that the spectrum of the operator D_0 is stable with respect to small perturbations of the form $V = iW^2$.

Corollary 3.1. *Let $m = 0$ and let $V = iW^2$ with $W^* = W$. Assume that*

$$(9) \quad \frac{1}{4} \int_{\mathbb{R}^2} \operatorname{tr}|V|dx < 1.$$

Then the operator $D = D_0 + V$ does not have eigenvalues outside of the real line \mathbb{R} , i.e. the spectrum of D is real.

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FRANCESCO FERRULLI, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, SW7 2AZ, LONDON, UK

E-mail address: f.ferrulli14@imperial.ac.uk

ARI LAPTEV, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, SW7 2AZ, LONDON, UK

E-mail address: laptev@mittag-leffler.se

OLEG SAFRONOV, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE, CHARLOTTE, NC 28223, USA

E-mail address: osafrono@uncc.edu